GALOIS LINES FOR NORMAL ELLIPTIC SPACE CURVES, III

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Abstract. We show the arrangement of V_4 and Z_4-lines for the linearly normal space elliptic curve with $j(E) = 1$. As a corollary, we show that each irreducible quartic curve with genus one has at most two Galois points.

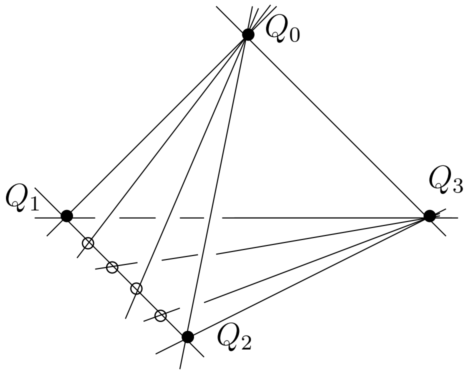
1. Introduction

We have been studying Galois embedding of algebraic varieties [5], in particular, of elliptic curves E. In this case, by Lemma 8 in [6] we can assume the embedding is associated with the complete linear system $|nP_0|$ for some $n \geq 3$, where $P_0 \in E$. Let $f_n : E \hookrightarrow \mathbb{P}^{n-1}$ be the embedding and put $C_n = f_n(E)$. Then we consider the Galois subspaces, Galois group, the arrangement of Galois subspaces and etc. for C_n in $\mathbb{P}^{n-1}$. In the previous papers [1, 6] we have treated in the case where $n = 4$ and settled almost all questions. However, the arrangement of V_4 and Z_4-lines has not been determined completely for $j(E) = 1$, i.e., the curve with an automorphism of order four with a fixed point. In this article we will complete it. Furthermore, we show the number of Galois points for an irreducible quartic curve of genus one, which is a correction of the assertion of Corollary 2 in [6].

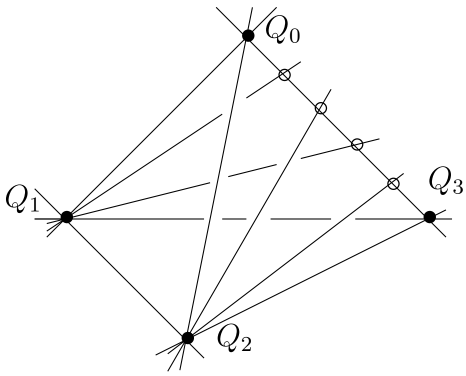
Key words and phrases. Galois point, genus-one curve, Galois group.

2. Statement of result

Theorem 1. *The arrangement of all the Galois lines for C_4, where $j(C_4) = 1$, is illustrated by the union of the following two figures:*



and



In these figures, $\bullet$ denotes the intersection of V_4-lines and $\circ$ denotes the intersection of a V_4-line and a Z_4-line. Four points Q_0, Q_1 Q_2 and Q_3 are not coplanar. These points form vertices of a tetrahedron. Let ℓ_{ij} be the line passing through Q_i and Q_j $(0 \le i < j \le 3)$. Then, all the V_4-lines are ℓ_{01}, ℓ_{02}, ℓ_{03}, ℓ_{12}, ℓ_{13} and ℓ_{23}. Except these lines, each line is a Z_4-line. For each vertex there exist two Z_4-lines passing through it. Two Z_4-lines which do not pass through the same vertex are disioint. A Z_4-line meets V_4-lines at two points. One is a vertex Q_i of the tetrahedron, we let the other be R_{ij} (which is indicated by $\circ$ in the figures), where $0 \le i \le 3$ and $j = 1,\ 2$. By taking a suitable coordinates of $\mathbb{P}^3$, we can give the coordinates of Q_i and R_{ij} explicitly as follows, in the following we use the notation $i = \sqrt{-1}$:

$Q_0 = (0:0:0:1),\ Q_1 = (4:-1:2:0),\ Q_2 = (4:-1:-2:0),$
$Q_3 = (4:1:0:0),$
$R_{01} = (0:0:1:0),\ R_{02} = (4:-1:0:0),\ R_{31} = (4:-1:2i:0),$
$R_{32} = (4:-1:-2i:0),\ R_{11} = (4:1:0:-2\sqrt{2}i),\ R_{12} = (4:1:0:2\sqrt{2}i),$

$R_{21} = (4 : 1 : 0 : 2\sqrt{2}),\ R_{22} = (4 : 1 : 0 : -2\sqrt{2})$

In Corollary 2 in [6] we must asume $j(E) \neq 1$. So we correct the corollary as follows:

Corollary 2. *Let Γ be an irreducible quartic curve in $\mathbb{P}^2$ and E the normalization of it. Assume the genus of E is one. If $j(E) = 1$ (resp. $\neq 1$), then the number of Galois points is at most two (resp. one).*

In fact, Takahashi found an example of such curves: $s^4 + s^2u^2 + t^4 = 0$. It is easy to see that the genus of the normalization is one and $(s : t : u) = (0 : 1 : 0)$ is a Z_4-point and $(1 : 0 : 0)$ is a V_4-point. We can find many such examples as follows:

Remark 1. Let L_{ij} and ℓ_{pq} be the Z_4 and V_4-lines passing though R_{ij}, where $0 \leq i \leq 3$, $j = 1, 2$ and if $i = 0$ or 3 (resp. 1 or 2), then $(p, q) = (1, 2)$ (resp. $(0, 3)$). Let $\pi_{ij} : \mathbb{P}^3 \cdots \longrightarrow \mathbb{P}^2$ be the projection with the center R_{ij}. Then, $\pi_{ij}(C_4) = \Gamma_{ij}$ is an irreducible quartic curve and the points $\pi_{ij}(L_{ij})$ and $\pi_{ij}(\ell_{pq})$ are Z_4 and V_4-points, respectively. For example, take the point $R = (0 : 0 : 1 : 0)$ as the projection center. Then, $\pi_R(X : Y : Z : W) = (X : Y : W)$. The Z_4-line $L : X = Y = 0$ and V_4-line $\ell : X + 4Y = W = 0$ pass through R. The defining equation of $\pi_R(C_4)$ is $W^4 = XY(X - 4Y)^2$, $\pi_R(L) = (0 : 0 : 1)$ and $\pi_R(\ell) = (-4 : 1 : 0)$. By the projective change of coordinates

$$X = X' - iY', \quad Y = -(X' + iY')/4$$

we get the example of Takahashi.

We have an interest in the group generated by the Galois groups associated with Galois lines [3]. In the current case we have the following:

Corollary 3. *Let $\mathcal{G}$ be the group generated by the groups associated with the Galois lines. Then, we have $\mathcal{G} \cong (Z_2 \times Z_4) \rtimes Z_4$.*

3. Proof

Hereafter we treat only the case $j(E) = 1$. We use the same notation and convention as in [6]. Let us recall briefly:

- $\pi : \mathbb{C} \longrightarrow E = \mathbb{C}/\mathcal{L}, \quad \mathcal{L} = \mathbb{Z} + \mathbb{Z}i, \quad i = \sqrt{-1}$
- $x = \wp(z),\ y = \wp'(z)$, $\wp$-functions with respect to $\mathcal{L}$.
- $\varphi : \mathbb{C} \longrightarrow \mathbb{C}/\mathcal{L} \xrightarrow{\sim} E : y^2 = 4x^3 - x$
- $P_\alpha := \varphi(\alpha) \in E$, $(\alpha \in \mathbb{C})$, in particular, $P_0 = \varphi(0)$
- $+$ denotes the sum of complex numbers $\alpha + \beta$ in $\mathbb{C}$ and at the same time the sum of divisors $P_\alpha + P_\beta$ on E
- $\sim$: linear equivalence
- Note that $P_\alpha + P_\beta \sim P_{\alpha+\beta} + P_0$ holds true.
- V_4 : Klein's four group
- Z_n : cyclic group of order n

- $\langle \cdots \rangle$: the group generated by $\cdots$

Since the embedding is associated with $|4P_0|$, we can assume it is given by

$$f = f_4 : E \longrightarrow \mathbb{P}^3, \quad f(x,y) = (1 : x^2 : x : y)$$

Put $C = f(E)$. The V_4-lines have been determined in [6]. Recall that the Galois group associated with V_4-line is $\langle \rho_i, \rho_j \rangle$ for some $i,\ j$ where $0 \le i < j \le 3$. Let σ be a complex representation of a generator of the group associated with Z_4-line. As we see in the proof of Lemma 20 in [6], σ can be expressed as $\sigma(z) = iz + (m+ni)/4$, where $(m,n) = (0,0),\ (2,2),\ (3,1),\ (1,3),\ (1,1),\ (3,3),\ (2,0)$ or $(0,2)$. So we put as follows:

(0) $\sigma_0(z) = iz$ (1) $\sigma_1(z) = iz + \dfrac{1+i}{2}$

(2) $\sigma_2(z) = iz + \dfrac{3+i}{4}$ (3) $\sigma_3(z) = iz + \dfrac{1+3i}{4}$

(4) $\sigma_4(z) = iz + \dfrac{1+i}{4}$ (5) $\sigma_5(z) = iz + \dfrac{3+3i}{4}$

(6) $\sigma_6(z) = iz + \dfrac{1}{2}$ (7) $\sigma_7(z) = iz + \dfrac{i}{2}$

Furthermore we put

$$\rho_0 = \sigma_0^2, \quad \rho_1 = \sigma_2^2, \quad \rho_2 = \sigma_4^2 \quad \rho_3 = \sigma_6^2 = \sigma_7^2.$$

Note that

$$\sigma_0^2 \equiv \sigma_1^2 (\mathrm{mod}\mathcal{L}),\ \sigma_2^2 \equiv \sigma_3^2 (\mathrm{mod}\mathcal{L}), \quad \sigma_4^2 \equiv \sigma_5^2 (\mathrm{mod}\mathcal{L}) \quad \sigma_6^2 \equiv \sigma_7^2 (\mathrm{mod}\mathcal{L}).$$

and

$$\rho_0(z) = -z, \quad \rho_1(z) = -z + \frac{1}{2}, \quad \rho_2(z) = -z + \frac{i}{2}, \quad \rho_3(z) = -z + \frac{1+i}{2}.$$

Let V be the vector space spanned by $\{1,\ x^2,\ x,\ y\}$ over $\mathbb{C}$. If σ is an element of the Galois group associated with a Galois line ℓ, then it induces a linear transformation $M(\sigma)$ of V. The $M(\sigma)$ defines a projective transformation, which is denoted by the same letter. It has the following properties:

(1) Some eigenvalue belongs to at least two independent eigenvectors.
(2) $M(\sigma)(\ell) = \ell$, i.e., $M(\ell)$ induces an automorphism of $\ell \cong \mathbb{P}^1$.

There are two characterizations of the vertices, one is the following Lemma 17 in [6]:

Lemma 1. *There exist exactly four irreducible quadratic surfaces S_i ($0 \le i \le 3$) such that each S_i has a singular point and contains C. Let Q_i be the unique singular point of S_i. Then the four points are not coplanar.*

The other one is as follows:

Lemma 2. *The $M(\rho_i)$ $(0 \le i \le 3)$ has two eigenvalues λ_{i1} and λ_{i2} which belong to one and three independent eigenvectors, respectively. Let Q_i be the point in $\mathbb{P}^3$ defined by the eigenvector having the eigenvalue λ_{i1}. Then, these points coincide with the ones in Lemma 1. Then the line passing through Q_i*

and Q_j $(0 \le i < j \le 3)$ *is a* V_4*-line. Four points* $\{Q_1,\ Q_2,\ Q_3,\ Q_4\}$ *are not coplanar, so they form a vertex of a tetrahedron* T*.*

Proof. These are checked by direct computations. To find the action of ρ_i on V, we have to find the one of ρ_i on $x = \wp(z)$ and $y = \wp'(z)$. For the purpose we use the addition formula on elliptic curve.

$$\begin{array}{lcl}
\rho_0(1, x^2, x, y) & = & (1, x^2, x, -y) \\
\rho_1(1, x^2, x, y) & = & (4x^2 - 4x + 1,\ x^2 + x + \frac{1}{4},\ 2x^2 - \frac{1}{2},\ 2y) \\
\rho_2(1, x^2, x, y) & = & (4x^2 + 4x + 1, x^2 - x + \frac{1}{4}, -2x^2 + \frac{1}{2},\ 2y) \\
\rho_3(1, x^2, x, y) & = & (16x^2, 1, -4x, -4y)
\end{array}$$

The representation matrices are as follows:

$$M(\rho_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad M(\rho_1) = \begin{pmatrix} 1 & 4 & -4 & 0 \\ \frac{1}{4} & 1 & 1 & 0 \\ -\frac{1}{2} & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$M(\rho_2) = \begin{pmatrix} 1 & 4 & 4 & 0 \\ \frac{1}{4} & 1 & -1 & 0 \\ \frac{1}{2} & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad M(\rho_3) = \begin{pmatrix} 0 & 16 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

Eigenvalues λ and eigenvectors of $M(\rho)$ are as foloows:
$M(\rho_0)$ $\lambda = -1 : (0, 0, 0, 1)$ $\lambda = 1 : (1, 0, 0, 0),\ (0, 1, 0, 0),\ (0, 0, 1, 0)$
$M(\rho_1)$ $\lambda = -2 : (4, -1, 2, 0)$ $\lambda = 2 : (1, 0, -1/2, 0),\ (0, 1, 1, 0),\ (0, 0, 0, 1)$
$M(\rho_2)$ $\lambda = -2 : (4, -1, -2, 0)$ $\lambda = 2 : (4, 0, 1, 0),\ (0, 1, -1, 0),\ (0, 0, 0, 1)$
$M(\rho_3)$ $\lambda = 4 : (4, 1, 0, 0)$ $\lambda = -4 : (4, -1, 0, 0),\ (0, 0, 1, 0),\ (0, 0, 0, 1)$

□

Similarly we can find Z_4-lines by the following calculations:

$$\begin{array}{rcl}
\sigma_0(1,x^2,x,y) &=& (1,x^2,-x,iy)\\
\sigma_1(1,x^2,x,y) &=& (16x^2,1,4x,4ix)\\
\sigma_2(1,x^2,x,y) &=& (-2y+\sqrt{2}(i-1)x^2-\sqrt{2}(1+i)x-\frac{\sqrt{2}(i-1)}{4},\\
&& -\frac{1}{2}y-\frac{\sqrt{2}(i-1)}{4}x^2+\frac{\sqrt{2}(i+1)}{4}x+\frac{\sqrt{2}(i-1)}{16},\\
&& \frac{\sqrt{2}+\sqrt{2}i}{2}x^2-\frac{\sqrt{2}i-\sqrt{2}}{2}x-\frac{\sqrt{2}+\sqrt{2}i}{8},2x^2+\frac{1}{2})\\
\sigma_3(1,x^2,x,y) &=& (4\sqrt{2}iy-(1+i)(4x^2+4ix-1),\ \frac{1}{4}(4\sqrt{2}iy+(1+i)(4x^2+4ix-1)),\\
&& \frac{i-1}{2}(4x^2-4ix-1),\ -\sqrt{2}(4x^2+1))\\
\sigma_4(1,x^2,x,y) &=& (-2\sqrt{2}(1+i)y-4ix^2-4x+i,\\
&& \frac{-1-i}{\sqrt{2}}y+ix^2+x-\frac{i}{4},\ 2x^2+2ix-\frac{1}{2},\\
&& -2\sqrt{2}(1+i)x^2-\frac{1+i}{\sqrt{2}})\\
\sigma_5(1,x^2,x,y) &=& (2\sqrt{2}(1+i)y-4ix^2-4x+i,\\
&& \frac{1+i}{\sqrt{2}y}+ix^2+x-\frac{i}{4},\ 2x^2+2ix-\frac{1}{2},\ -\frac{1+i}{\sqrt{2}}(4x^2+1))\\
\sigma_6(1,x^2,x,y) &=& (4x^2+4x+1,x^2-x+\frac{1}{4},2x^2-\frac{1}{2},-2iy)\\
\sigma_7(1,x^2,x,y) &=& (4x^2-4x+1,x^2x+\frac{1}{4},-2x^2+\frac{1}{2},-2iy)
\end{array}$$

$$M(\sigma_0)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&i\end{pmatrix}\quad M(\sigma_1)=\begin{pmatrix}0&4&0&0\\\frac{1}{4}&0&0&0\\0&0&1&0\\0&0&0&i\end{pmatrix}$$

$$M(\sigma_2)=\begin{pmatrix}\frac{i+1}{4}&-i-1&1-i&-\sqrt{2}i\\-\frac{i+1}{16}&\frac{i+1}{4}&\frac{i-1}{4}&-\frac{i}{2\sqrt{2}}\\-\frac{i-1}{8}&\frac{i-1}{2}&\frac{i+1}{2}&0\\\frac{i}{2\sqrt{2}}&\sqrt{2}i&0&0\end{pmatrix}$$

$$M(\sigma_3)=\begin{pmatrix}1+i&-4(1+i)&4(1-i)&4\sqrt{2}i\\-\frac{1+i}{4}&1+i&i-1&\sqrt{2}i\\\frac{1-i}{2}&-2(1-i)&2(1+i)&0\\-\sqrt{2}i&-4\sqrt{2}&0&0\end{pmatrix}$$

$$M(\sigma_4)=\begin{pmatrix}i&-4i&-4&-2\sqrt{2}(1+i)\\-\frac{i}{4}&i&1&-\frac{1+i}{\sqrt{2}}\\-\frac{1}{2}&2&2i&0\\-\frac{1+i}{\sqrt{2}}&-2\sqrt{2}(1+i)&0&0\end{pmatrix}$$

$$M(\sigma_5)=\begin{pmatrix}i&-4i&-4&2v(1+i)\\-\frac{i}{4}&i&1&\frac{1+i}{\sqrt{2}}\\-\frac{1}{2}&2&2i&0\\\frac{1+i}{\sqrt{2}}&2\sqrt{2}(1+i)&0&0\end{pmatrix}$$

$$M(\sigma_6)=\begin{pmatrix}1&4&4&0\\\frac{1}{4}&1&-1&0\\-\frac{1}{2}&2&0&0\\0&0&0&-2i\end{pmatrix}\quad M(\sigma_7)=\begin{pmatrix}1&4&-4&0\\\frac{1}{4}&1&1&0\\\frac{1}{2}&-2&0&0\\0&0&0&-2i\end{pmatrix}$$

Eigenvalues λ and eigenvectors of $M(\sigma)$ are as follows:
$M(\sigma_0)$ $\lambda = -1 : (4, -1, 0, 0)$ $\lambda = 1 : (4, 1, 0, 0),\ (0, 0, 1, 0), \lambda = i\ (0, 0, 0, 1)$
$M(\sigma_1)$ $\lambda = -1 : (4, -1, 0, 0)$ $\lambda = 1 : (4, 1, 0, 0),\ (0, 0, 1, 0), \lambda = i\ (0, 0, 0, 1)$
$M(\sigma_2)$ $\lambda = i : (4, -1, -2, 0)$ $\lambda = 1 : (4\sqrt{2}, 0, \sqrt{2}, 2i),\ (0, 1, -1, \sqrt{2}i), \lambda = -1\ (4\sqrt{2}, \sqrt{2}, 0, -4i)$
$M(\sigma_3)$ $\lambda = i : (4, -1, -2, 0)$ $\lambda = 1 : (4\sqrt{2}, 0, \sqrt{2}, -2i),\ (0, 1, -1, -\sqrt{2}i), \lambda = -1\ (4\sqrt{2}, \sqrt{2}, 0, 4i)$
$M(\sigma_4)$ $\lambda = -2-2i : (4\sqrt{2}, \sqrt{2}, 0, 4)$ $\lambda = 2+2i : (4\sqrt{2}, 0, -\sqrt{2}, -2),\ (0, 1, 1, -\sqrt{2}), \lambda = -2 + 2i\ (4, -1, 2, 0)$
$M(\sigma_5)$ $\lambda = -2-2i : (4\sqrt{2}, \sqrt{2}, 0, -4)$ $\lambda = 2+2i : (4\sqrt{2}, 0, -\sqrt{2}, 2),\ (0, 1, 1, \sqrt{2}), \lambda = -2 + 2i\ (4, -1, 2, 0)$
$M(\sigma_6)$ $\lambda = 2i : (4, -1, 2i, 0)$ $\lambda = -2i : (4, -1, -2, 0),\ (0, 0, 0, 1), \lambda = 2\ (4, 1, 0, 0)$
$M(\sigma_7)$ $\lambda = 2i : (4, -1, -2i, 0)$ $\lambda = -2i : (4, -1, -2i, 0),\ (0, 0, 0, 1), \lambda = 2\ (4, 1, 0, 0)$

Corollary 4. (1) *In case* $J \neq 12^3$, $\mathcal{G}_0 = \langle \rho_0, \rho_1, \rho_2 \rangle \cong Z_2 \times Z_2 \times Z_2$. *an example of the curve with this group is given in* [4]

$$(4y^4 + 5xy^2 - 1)^2 = xy^2(x + 8y^2)^2.$$

(2) *In cse* $J = 12^3$ *we can show* $\mathcal{G} = \langle \sigma_0, \sigma_2, \sigma_6 \rangle$. *Putting*

$$\alpha(z) = z + \frac{1}{2}, \quad \beta(z) = z + \frac{1+i}{4},$$

we have $\langle \alpha, \beta \rangle \cong Z_2 \times Z_4$ *and*

$$\mathcal{G} \cong \langle \alpha, \beta \rangle \rtimes \langle \sigma_0 \rangle$$

It is easy to see that $\mathcal{G}_0$ *is a normal subgroup of* $\mathcal{G}$. *In particular* $|\mathcal{G}| = 32$ *and* $\mathcal{G}$ *is called an elliptic exceptional group* $E(2, 2, 4)$ *in* [4]. *Furthermore this group is appear as the group by the embedding of degree 32 of the elliptic curve* $j(E) = 1$.

References

1. C. Duyaguit and H. Yoshihara, Galois lines for normal elliptic space curvea, *Algebra Colloquium*, **12** (2005), 205–212.
2. J. Harris, Galois groups of enumerative problems, *Duke Math. J.*, **46** (1979), 685–724.
3. M. Kanazawa, T. Takahashi and H. Yoshihara, The group generated by automorphism belonging to Galois points of the quartic surface, *Nihonkai Math. J.*, **12** (2001), 89–99.
4. M. Kanazawa and H. Yoshihara, Galois group at Galois point for genus-one curve, *Int. J. Algebra*, **5** (2011), 1161–1174.
5. H. Yoshihara, Galois embedding of algebraic variety and its application to abelian surface, *Rend. Sem. Mat. Univ. Padova*, **117** (2007), 69–86
6. H. Yoshihara, Galois lines for normal elliptic space curves, II, *Algebra Colloquium*, **19** (2012), no.spec 01, 867–876

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